J. Fluid Mech. (1972), *vol.* 55, *part* 1, *pp.* 49–63
Printed in Great Britain

Steady two-dimensional viscous flow in a jet

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(Received 28 September 1971 and in revised form 12 June 1972)

An idealized two-dimensional flow due to a point source of x momentum is discussed. In the far field the flow is modelled by a jet region of large vorticity outside which the flow is potential. After use of the transformation

$$\zeta^3 = (\xi + i\eta)^3 = x + iy,$$

the equations suggest naively obvious asymptotic expansions for the stream function in these two regions, namely

$$\sum_{n=0}^{\infty} \xi^{1-n} f_n(\eta) \quad \text{and} \quad \sum_{n=0}^{\infty} \xi^{1-n} F_n(\eta/\xi)$$

respectively. Consistency in matching these expansions is achieved by including logarithmic terms associated with the occurrence of eigensolutions. F_n is easy to find and f_n can be found in closed form so the inner and outer eigensolutions may be fully determined along with the complete structure of the expansions.

1. Introduction

The problem is that of steady two-dimensional flow due to a point source of x^* momentum at the origin O. (In §§ 1 and 2 asterisks distinguish physical quantities from their non-dimensional counterparts.) The fluid is viscous, incompressible and of infinite extent. The flow is produced solely by the momentum source; there are no other singularities such as mass sources and there are no boundaries.

In the corresponding three-dimensional Landau source problem (Rosenhead 1963, p. 150) an exact self-similar solution has been found to describe the whole flow field. The two-dimensional case does not lend itself to an exact analytical description as does the three-dimensional case, and the method of matched asymptotic expansions is used to discuss the far flow field in the ζ plane, where $\zeta^3 = (\xi + i\eta)^3 = x + iy$. An inner stream function expansion $\sum_{n=0}^{\infty} \xi^{1-n} f_n(\eta)$ fails to match an outer expansion $\sum_{n=0}^{\infty} \xi^{1-n} F_n(\eta/\xi)$ beyond the third term. At the very root of such difficulties is the use of similarity solutions to give asymptotic descriptions of the inner and outer flows. The details of the near flow field do not influence such solutions except through an insistence on flow symmetry and the use of a flux integral condition which expresses conservation of momentum. Mathematically, the partial differential equations for the inner or jet region are replaced by ordinary differential equations with boundary conditions specified at $\eta = 0$ and $\eta = \infty$. The price of this simplification is the loss of an upstream boundary condition at $\xi = \xi_1 > 0$ which would otherwise have been applied to the essentially parabolic boundary-layer equations. Likewise, in the outer flow problem, boundary conditions that would otherwise accompany the potential equation are ignored. Thus one eigenfunction problem arises for the inner region and another for the outer. The final expansions contain arbitrary multiples of eigensolutions and can be used to describe far flow fields of more general force distributions than the point force actually considered. However, such distributions must have symmetry about Ox and exhibit behaviour at infinity no worse than that of the point force.

The problem is related to a certain two-dimensional jet problem considered first by Schlichting (1960, p. 164). He considered the case of laminar flow in a jet emerging from a slit in an infinite wall and, using boundary-layer theory, obtained a numerical solution for the jet region downstream of the slit. Bickley (1937) found the stream function similarity solution in a simple form $1{\cdot}6510(M\nu x^*)^{\frac{1}{3}} \tanh \eta_0$, where the similarity variable η_0 is $0{\cdot}2752(M/\nu^2)^{\frac{1}{3}} y^*/x^{*\frac{2}{3}}$. Here ρM is the source strength of x^* momentum while ν is the kinematic viscosity. It is not surprising that the boundary-layer solution in our problem is in effect the same as Bickley's, for his result depends on neither the wall geometry nor the rate of mass injection at the slit. Rubin & Falco (1968) found the dominant potential flow for the wall geometry considered by Bickley. They also found the first correction term for the jet region and made a brief study of the inner eigenvalue problem. This eigenvalue problem is independent of both the wall geometry and the rate of mass injection and thus is precisely the one to be considered here. The eigenfunction problem may be solved exactly; the results are given in §11.

In this paper a kinematic approach is adopted because for this problem it is simpler than the conventional dynamical one. In the exact problem, instead of considering a vector velocity field and a pressure field as in the dynamical approach, we simply require two scalar fields, the stream function and the vorticity (or circulation density). Locally, the transport of momentum depends not only on diffusion and convection but also on the pressure distribution, whereas transport of circulation is by diffusion and convection alone; detailed local comparison of the two terms in the circulation flux vector gives useful information about the physical structure of the flow field (Pillow 1970). Moreover, the dynamical approach affords no compensating advantage.

The kinematic approach leads us naturally to seek a kinematic description of the singularity at O. This description is quite simple and, see §2, provides the dominant vorticity field near the singularity by inspection. No such simplification occurs in the dynamical approach. Again it is the physical processes governing circulation transport that are important in establishing the nature of the far flow field.

In dynamical terms the flow is caused by a (two-dimensional) point force at O producing $\frac{16}{9}\rho M$ units of x^* momentum per unit time. Here ρ is the density and the factor $\frac{16}{9}$ is introduced to simplify subsequent expressions. Under steady conditions, this point force is equivalent to a dipole producing $\frac{16}{9}M$ units of circulation moment about Ox^* per unit time, where the circulation moment within a region S is defined as $\iint_S \omega y\, dS$ (see appendix). Thus the dipole has strength $\frac{16}{9}M$ and its axis is in the positive-y^* direction. Conservation of x^* momentum implies conservation of moment of circulation about Ox^* and vice versa.

2. Statement of the problem

Dimensionless variables ω, ψ, r and θ are defined by

$$\omega^* = M^2\nu^{-3}\omega, \quad \psi^* = \nu\psi, \quad r^* = \nu^2 M^{-1} r, \quad \theta^* = \theta,$$

while x, y, s are defined in the obvious way. A non-dimensional statement of the problem in polar co-ordinates is

$$\nabla^2\omega + \frac{1}{r}\frac{\partial(\psi,\omega)}{\partial(r,\theta)} = 0, \tag{2.1}$$

$$\nabla^2\psi + \omega = 0, \tag{2.2}$$

$$\oint_C \mathbf{n}.\mathbf{V}\, ds = \tfrac{16}{9}, \tag{2.3}$$

$$\psi = \omega = 0 \quad \text{on} \quad \theta = 0, \pm\pi, \tag{2.4}$$

where $r > 0$, $-\pi \leqslant \theta \leqslant \pi$ and ∇^2 is the Laplace operator. The conservation result (2.3) is derived in the appendix. C is any simple closed contour enclosing O, with outward unit normal $\mathbf{n}$ and arc length s; $\mathbf{V}$ is a suitable flux vector for the moment of circulation about Ox. When C is a circle with centre O and radius r, (2.3) is replaced by

$$\int_{-\pi}^{\pi} V_1 r\, d\theta = \tfrac{16}{9}, \tag{2.5}$$

where V_1 is the radial component of $\mathbf{V}$.

Since a simple solution of (2.1)–(2.5) appears unlikely and since there are no natural parameters in the equation, we consider asymptotic solutions for small and large values of r, which acts as a 'local Reynolds number'. Thus $Mr^*/\nu^2 \gg 1$ in the latter case, which is the one of main interest here. Now for r sufficiently small, diffusion effects dominate convection effects so that ω is approximately harmonic and ψ approximately biharmonic. As leading terms in a co-ordinate perturbation expansion for ω and ψ we may take $(2\pi r)^{-1}\sin\theta$ and

$$-(4\pi)^{-1} r \ln r \sin\theta,$$

respectively. The flow for $r \gg 1$, $-\pi \leqslant \theta \leqslant \pi$ is conveniently discussed by introducing the transformation

$$re^{i\theta} = z = x + iy = (\xi + i\eta)^3 = \zeta^3 = (Re^{i\phi})^3. \tag{2.6}$$

The exact problem (2.1)–(2.4) becomes, for the transformed domain $\xi > 0$, $|\eta| \leqslant \xi\sqrt{3}$,

$$\nabla^2\omega + \partial(\psi,\omega)/\partial(\xi,\eta) = 0, \tag{2.7}$$

$$\nabla^2\psi + 9(\xi^2+\eta^2)^2\omega = 0, \tag{2.8}$$

$$\int_\Gamma V_\xi\, d\eta - V_\eta\, d\xi = \tfrac{16}{27}, \tag{2.9}$$

$$\psi = \omega = 0 \quad \text{on} \quad \eta = 0, \pm\xi\sqrt{3}. \tag{2.10}$$

Here $\nabla^2 = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$ and Γ is the image of C under the transformation. When Γ is that part of the straight line $\xi = \alpha$ (α a positive constant) between $\eta = \mp\xi\sqrt{3}$, (2.9) becomes

$$\int_0^{\xi\sqrt{3}} V_\xi \, d\eta = \tfrac{8}{27}, \tag{2.11}$$

after using the flow symmetry. We need record only one flux vector component:

$$V_\xi = \left[-\tfrac{1}{3}\eta(3\xi^2-\eta^2)\frac{\partial\omega}{\partial\xi} + 2\xi\eta\omega + \tfrac{1}{3}\eta\omega(3\xi^2-\eta^2)\frac{\partial\psi}{\partial\eta}\right] + \frac{1}{18(\xi^2+\eta^2)^2}\left[(\eta^2-\xi^2)\left(\frac{\partial\psi}{\partial\xi}\right)^2 + 4\xi\eta\frac{\partial\psi}{\partial\xi}\frac{\partial\psi}{\partial\eta} + (\xi^2-\eta^2)\left(\frac{\partial\psi}{\partial\eta}\right)^2\right]. \tag{2.12}$$

3. The boundary-layer solution

For $r \gg 1$, we can hardly expect a regular perturbation problem. The convection effects cause the circulation to be carried downstream along the streamlines while the viscous effects cause it to diffuse like heat as it is being convected. Furthermore, no circulation is convected from 'upstream infinity', so that its appearance other than in a region downstream of O must be due to the diffusion process and its density ω is thus expected to be exponentially small far from the axis. Moreover, a boundary-layer region is observed in experimental two-dimensional jets. The flow is modelled by a region of large vorticity outside which the flow is potential: we are faced with a singular perturbation problem.

A solution for $r \gg 1$, $-\pi \leqslant \theta \leqslant \pi$ corresponds to a solution for $R \gg 1$, $-\frac{1}{3}\pi \leqslant \phi \leqslant \frac{1}{3}\pi$ or, equivalently, $\xi \gg 1$, $-\xi\sqrt{3} \leqslant \eta \leqslant \xi\sqrt{3}$ in the ζ plane. In the boundary-layer approximation, $\partial^2\omega/\partial\xi^2$ and $\partial^2\psi/\partial\xi^2$ in (2.7) and (2.8) are insignificant while the equally dominant terms in V_ξ are $\omega\eta\xi^2\,\partial\psi/\partial\eta$ and $\frac{1}{18}\xi^{-2}(\partial\psi/\partial\eta)^2$. The boundary-layer solution has the form $\psi_0 = \xi f_0(\eta)$, where

$$f_0^{\mathrm{iv}} + f_0 f_0''' + 3f_0' f_0'' = 0, \tag{3.1}$$

$$\int_0^\infty (f_0'^2 - 2\eta f_0' f_0'')\, d\eta = \tfrac{16}{3}, \tag{3.2}$$

$$f_0(0) = f_0''(0) = 0, \quad f_0^{(n)}(\infty) = 0 \quad (n = 1, 2, \ldots). \tag{3.3}$$

The conditions at $\eta = 0$ describe the flow symmetry while those at infinity arise from the decay of vorticity in the lateral direction and the quiescence of the surrounding fluid. Because of the flow symmetry we need consider the situation for $\eta \geqslant 0$ only. The solution of (3.1)–(3.3) is

$$f_0 = 2\tanh\eta, \tag{3.4}$$

so that

$$\psi_0 = 2\xi\tanh\eta, \tag{3.5}$$

which is essentially the result found by Bickley, the inner similarity variable η being approximately proportional to Bickley's η_0. The boundary-layer streamlines for $\xi \gg 1$ are very nearly the straight lines $\eta = $ constant.

4. Inner and outer expansions

In the inner or jet region, we are concerned with limits as $\xi \to \infty$ while η remains fixed; $\eta = o(\xi)$ in this region. In the outer region $\eta = O(\xi)$ and here we are concerned with limits as $\xi \to \infty$ while $\chi = \eta/\xi$ remains fixed. Thus χ is a similarity variable for the outer region.

For the inner region consider expansions of the form

$$\begin{aligned}\psi(\xi,\eta) &= \psi_0(\xi,\eta)+\psi_1(\xi,\eta)+\ldots+\psi_n(\xi,\eta)+\ldots\\ &= \xi f_0(\eta)+h_1(\xi)f_1(\eta)+\ldots+h_n(\xi)f_n(\eta)+\ldots, \end{aligned} \tag{4.1}$$

$$\begin{aligned}\omega(\xi,\eta) &= \omega_0(\xi,\eta)+\omega_1(\xi,\eta)+\ldots+\omega_n(\xi,\eta)+\ldots\\ &= \xi^{-3}g_0(\eta)+\xi^{-4}h_1(\xi)\,g_1(\eta)+\ldots+\xi^{-4}h_n(\xi)\,g_n(\eta)+\ldots, \end{aligned} \tag{4.2}$$

where $h_{n+1}/h_n \to 0$ as $\xi \to \infty$. These must match with outer expansions:

$$\begin{aligned}\psi(\xi,\eta) &= \Psi_0(\xi,\eta)+\Psi_1(\xi,\eta)+\ldots+\Psi_n(\xi,\eta)+\ldots\\ &= \xi F_0(\chi)+h_1(\xi)\,F_1(\chi)+\ldots+h_n(\xi)\,F_n(\chi)+\ldots, \end{aligned} \tag{4.3}$$

$$\omega(\xi,\eta) = \Omega_0(\xi,\eta)+\Omega_1(\xi,\eta)+\ldots+\Omega_n(\xi,\eta)+\ldots. \tag{4.4}$$

The reasonable assumption of exponentially small vorticity at 'upstream infinity' coupled with an induction argument shows that $\Omega_n \equiv 0$ for $n = 0, 1, \ldots$. First, from the terms of highest order in (2.7), after substitution of (4.3) and (4.4), we have

$$\partial(\Psi_0, \Omega_0)/\partial(\xi,\eta) = 0.$$

Thus Ω_0 is constant on any streamline. The assumption of exponential decay of vorticity at 'upstream infinity' then implies $\Omega_0 \equiv 0$. Suppose now that $\Omega_i = 0$ for $i = 1, 2, \ldots, n-1$. Then the equation for Ω_n, namely,

$$\partial(\Psi_0, \Omega_n)/\partial(\xi,\eta) = 0,$$

implies in a similar way that $\Omega_n \equiv 0$. By induction, all terms of (4.4) are zero so that the terms Ψ_n are harmonic.

Now the integral condition (2.9), retained in full but exhibiting the inner and outer contributions, is

$$\int_0^{\eta_L} V_\xi\, d\eta + \int_{\chi_L}^{\sqrt{3}} V_\xi\, \xi\, d\chi = \tfrac{8}{27}, \tag{4.5}$$

where η_L and χ_L are values of η and χ at a point L in the region of overlap. (Under an appropriate limit process, $\eta_L \to \infty$ and $\chi_L \to 0$.) Careful inspection of (2.7), (2.8) and (4.5) after using (2.12), (4.1), (4.2) and (4.3) shows that, at least temporarily, we should write

$$\psi(\xi,\eta) = \xi f_0(\eta)+f_1(\eta)+\xi^{-1}f_2(\eta)+\ldots, \tag{4.6}$$

$$\omega(\xi,\eta) = \xi^{-3}g_0(\eta)+\xi^{-4}g_1(\eta)+\xi^{-5}g_2(\eta)+\ldots \tag{4.7}$$

for the inner region, and for the outer region

$$\psi(\xi,\eta) = \xi F_0(\chi)+F_1(\chi)+\xi^{-1}F_2(\chi)+\ldots. \tag{4.8}$$

In addition to the flux condition (4.5), we have the conditions at $\eta = 0$ in

(2.10), which may be applied to the inner expansions (4.6) and (4.7), and the conditions at $\eta = \pm\xi\sqrt{3}$, which may be applied to (4.8). It is necessary that the inner flow should merge into the correct potential flow with exponential decay of vorticity as $\eta \to \infty$.

5. Potential flow induced by the basic jet flow

The fluid on either side of the jet is set in motion by the jet action. Now $\psi_0 \to 2\xi$ as $\eta \to \infty$. This limit gives the order of the potential flow and is indeed itself harmonic; however, the harmonic function required ($\Psi_0 = \xi F_0(\chi)$) must vanish on $\eta = \xi\sqrt{3}$. This function is given uniquely by

$$\Psi_0 = 2\xi(1-\chi/\sqrt{3}) = 2(\xi-\eta/\sqrt{3}). \tag{5.1}$$

For the lower half of the potential-flow region $\Psi_0 = -2(\xi+\eta/\sqrt{3})$, which vanishes on $\eta = -\xi\sqrt{3}$ and matches ψ_0 as $\eta \to -\infty$. The potential flow Ψ_0 in the z plane is completely smooth at $\theta = \pi$. The most convenient descriptions of the flow terms in the z plane are obtained using the range $0 \leqslant \theta < 2\pi$; thus

$$\Psi_0 = (4/\sqrt{3})\, r^{\frac{1}{3}} \sin \tfrac{1}{3}(\pi-\theta) \quad (0 < \theta < 2\pi). \tag{5.2}$$

The discontinuity $-\frac{4}{3}r^{-\frac{2}{3}}$ in the θ component of velocity on traversing the jet from below may be interpreted as arising from a line distribution of sinks along the positive x axis, the sink density being proportional to $x^{-\frac{2}{3}}$. To leading order, this describes the entrainment effect of the jet on the surrounding fluid.

6. General term of the inner expansion

Substitution of (4.6) and (4.7) in (2.7) and (2.8) yields (from the coefficients of ξ^{-3-n} and ξ^{1-n} in these respective equations)

$$g_n'' + f_0 g_n' + (n+3) f_0' g_n + 3 g_0 f_n' - (n-1) g_0' f_n = H_n, \tag{6.1}$$

$$f_n'' + 9 g_n = K_n, \tag{6.2}$$

where
$$K_n = -[(3-n)(2-n) f_{n-2} + 18\eta^2 g_{n-2} + 9\eta^4 g_{n-4}], \tag{6.3}$$

while H_n is a function of η with terms containing $f_{n-1}, g_{n-1}, \ldots, f_1, g_1$ and their derivatives. These equations are strictly correct only for $n \geqslant 4$; they are correct for $n = 2$ and 3 provided that we write $g_{n-4} = 0$ for these values. The correct equations for $n = 1$ are obtained by writing $H_1 = K_1 = 0$. Elimination of the g_i ($i = 0, 1, \ldots, n$) from (6.1) and (6.2) yields for $n \geqslant 1$

$$M_n(f_n) = s_n, \tag{6.4}$$

where the operator M_n is given by

$$M_n = \frac{d^4}{d\eta^4} + f_0 \frac{d^3}{d\eta^3} + (n+3) f_0' \frac{d^2}{d\eta^2} + 3 f_0'' \frac{d}{d\eta} - (n-1) f_0''', \tag{6.5}$$

and the forcing term s_n may be expressed in terms of the f_i ($i = 0, 1, \ldots, n-1$) and their derivatives. The boundary conditions are

$$f_n(0) = f_n''(0) = 0, \quad g_n(\eta) \to 0 \quad \text{exponentially as} \quad \eta \to \infty. \tag{6.6}$$

Then (6.4) implies that f_n is an odd function. Integration of (6.4) yields

$$L_n(f_n) = S_n, \tag{6.7}$$

where
$$L_n = \frac{d^3}{d\eta^3}+f_0\frac{d^2}{d\eta^2}+(n+2)f_0'\frac{d}{d\eta}-(n-1)f_0'' \tag{6.8}$$

and
$$S_n = B_n+\int_0^\eta s_n(\bar\eta)\,d\bar\eta, \quad B_n = \text{constant}. \tag{6.9}$$

Since $L_n(f_0') = 0$, the substitution $f_n = uf_0'$ reduces (6.7) to

$$f_0'U''+2f_0''U'+(f_0'''+nf_0'^2)\,U = S_n,$$

where $U = u'$; this equation has been simplified by making use of the integrated forms of (3.1). By writing $X = f_0'U$ and changing the independent variable to

$$\tau = \tau(\eta) = \tanh\eta = \tfrac{1}{2}f_0,$$

we obtain
$$(1-\tau^2)\frac{d^2X}{d\tau^2}-2\tau\frac{dX}{d\tau}+\nu(\nu+1)\,X = 2S_n/f_0', \tag{6.10}$$

where $\nu(\nu+1) = 2n$. Three fundamental solutions of (6.7) and their behaviour as $\eta\to\infty$ are

$$u_1 = f_0' \to 0 \text{ exponentially}, \tag{6.11}$$

$$u_2 = f_0'\int_0^\eta \frac{P_\nu[\tau(\bar\eta)]\,d\bar\eta}{f_0'(\bar\eta)} \sim \text{constant}+O(e^{-2\eta}), \tag{6.12}$$

$$u_3 = f_0'\int_0^\eta \frac{Q_\nu[\tau(\bar\eta)]\,d\bar\eta}{f_0'(\bar\eta)} \sim \text{constant}\times\eta+\text{constant}+O(e^{-2\eta}), \tag{6.13}$$

where P_ν and Q_ν are Legendre functions of the first and second kind. A fourth fundamental solution of (6.4) generated by B_n is

$$u_4 = f_0'\int_0^\eta \frac{\{\alpha_n P_\nu[\tau(\bar\eta)]+\beta_n Q_\nu[\tau(\bar\eta)]\}\,d\bar\eta}{f_0'(\bar\eta)}, \tag{6.14}$$

where
$$\alpha_n = -\int_0^{\tau(\bar\eta)}\frac{Q_\nu(\bar\tau)\,d\bar\tau}{1-\bar\tau^2}, \quad \beta_n = \int_0^{\tau(\bar\eta)}\frac{P_\nu(\bar\tau)\,d\bar\tau}{1-\bar\tau^2}.$$

For $n = 1, 2, \ldots, u_4$ asymptotes to a quadratic polynomial in η with exponentially small error for large η. The form of (6.7) implies that, since the forcing term S_n is even, the particular integral W_n to which it gives rise is odd. By (6.6) we require two odd fundamental solutions. For $n = 1, 2, \ldots$, two such solutions are u_4 and $a_2u_2+a_3u_3$, where

$$a_2u_2''(0)+a_3u_3''(0) = 0,$$

which reduces to
$$2a_2\sin\tfrac{1}{2}\nu\pi+\pi a_3\cos\tfrac{1}{2}\nu\pi = 0. \tag{6.15}$$

As the second odd fundamental solution we select

$$\left.\begin{array}{l} u_2, \text{ when } \nu \text{ is an even integer and } n = k(2k+1),\ k = 1, 2, \ldots, \\ u_3, \text{ when } \nu \text{ is an odd integer and } n = k(2k-1),\ k = 1, 2, \ldots, \\ u_2-(2/\pi)\tan\tfrac{1}{2}\nu\pi\, u_3, \text{ when } \nu \text{ is not an integer.} \end{array}\right\} \tag{6.16}$$

As fundamental solutions we shall use suitable linear combinations

$$u_{ni}\ (i = 1,2,3,4) \quad \text{of} \quad u_j\ (j = 1,2,3,4).$$

Although it is possible to construct a particular integral W_n, it is more convenient to discuss each case in turn.

7. General term of the outer expansion

The general term of (4.8) may be written as $R^{1-n}\Phi_n(\phi)$, where

$$\Phi_n = \cos^{1-n}\phi F_n(\tan\phi).$$

The general solution of Laplace's equation which vanishes on $\phi = \frac{1}{3}\pi$ is an arbitrary multiple of $(\frac{1}{3}\pi-\phi)$ or $R^{1-n}\sin[(n-1)(\frac{1}{3}\pi-\phi)]$ according as $n = 1$ or $n \neq 1$. These correspond to solutions $\frac{1}{3}(\pi-\theta)$ and $r^{\frac{1}{3}(1-n)}\sin[\frac{1}{3}(n-1)(\pi-\theta)]$ in the z plane. The constant multipliers are found by matching with (4.6) after expanding the potential solutions about $\chi = 0$. Later, a complex representation of these potential terms is given.

8. Solution for f_1 and F_1

When $n = 1$, the particular integral of (6.4) is zero since $s_1 = 0$. Moreover, $\nu = 1$ and the three fundamental solutions have simple forms:

$$u_{11} = f_0' \text{ (even)}, \quad u_{12} = 1 \text{ (even)},$$
$$u_{13} = (1-3\tau^2)-3\tau \text{ (odd)} \sim -2\eta+3 \quad \text{for} \quad \eta \gg 1.$$

Again $u_{14} = u_4$ with $\nu = 1$ and this odd solution asymptotes to a quadratic for large η. Now for $\eta \gg 1$, $\xi^0 f_1$ must match the potential terms of order ξ^0 for $\eta/\xi \ll 1$. Inspection of (5.1) shows that u_{14} must be excluded and further that

$$\psi_1 = f_1 = u_{13}/\sqrt{3} = [\eta(1-3\tau^2)+3\tau]/\sqrt{3}. \tag{8.1}$$

In its turn, the asymptotic behaviour of f_1 shows that in the outer solution we require a term which behaves like $\xi^0\sqrt{3}$ for $\eta/\xi \ll 1$. The required potential term $\xi^0 F_1(\chi)$ is

$$\Psi_1 = \sqrt{3}-3\phi\sqrt{3}/\pi = [\pi-3\arctan(\eta/\xi)]\sqrt{3}/\pi, \tag{8.2}$$

since this vanishes on $\eta = \xi\sqrt{3}$. The results may be extended to the lower half field as before. In the z plane, the potential flow of this order is

$$\Psi_1 = \sqrt{3}(\pi-\theta)/\pi \quad (0 < \theta < 2\pi), \tag{8.3}$$

which signifies a sink flow towards the origin. Such a flow independent of r is acceptable on physical grounds since fluid driven downstream by the point force has to be replaced. Because Ψ_0 has a particularly simple form, it is Ψ_1 which dictates the dominant behaviour for $\eta \gg 1$ of subsequent terms $\xi^{-1}f_2, \xi^{-2}f_3, \ldots$ in the inner expansion. For $\eta/\xi \ll 1$,

$$\Psi_1 = \sqrt{3}-\frac{3\sqrt{3}}{\pi}\xi^{-1}\eta+\frac{\sqrt{3}}{\pi}\xi^{-3}\eta^3-\ldots.$$

The matching scheme in table 1 extends beyond the terms discussed in this paper.

For $\eta/\xi \ll 1$, expansion of	ξ	ξ^{0}	ξ^{-1}	$\xi^{-2}\log\xi$	ξ^{-2}	$\xi^{-3}\log\xi$	ξ^{-3}	$\xi^{-4}(\log\xi)^{2}$	$\xi^{-4}\log\xi$	ξ^{-4}	
	Order of terms										
$\Psi_0 =$	2ξ	$-\dfrac{2\eta}{\sqrt{3}}$									
$\Psi_1 =$		$\sqrt{3}$	$-\dfrac{3\sqrt{3}\eta}{\pi\xi}$				$+\dfrac{\sqrt{3}\eta^{3}}{\pi\xi^{3}}$				$+\dots$
$\Psi_2 =$			$\dfrac{4{\cdot}1262}{\xi}$		$-\dfrac{4{\cdot}1262\eta}{\sqrt{3}\,\xi^{2}}$		$-\dfrac{4{\cdot}1262\eta^{2}}{\xi^{3}}$			$+\dfrac{4{\cdot}1262\eta^{3}}{\sqrt{3}\xi^{4}}$	$+\dots$
$\Psi_3 =$				$\dfrac{\sqrt{3}B\ln\xi}{2\xi^{2}}$	$+\left(\dfrac{\pi B}{6}+\dfrac{\sqrt{3}A}{2}\right)\dfrac{1}{\xi^{2}}$	$+\dfrac{B\eta\ln\xi}{\xi^{3}}$	$+\left(\dfrac{B}{2}-\dfrac{\sqrt{3}\pi B}{3}+A\right)\dfrac{\eta}{\xi^{3}}$		$-\dfrac{3\sqrt{3}B}{2}\,\dfrac{\eta^{2}\ln\xi}{\xi^{4}}$	$-\left(\dfrac{5\sqrt{3}B}{4}-\dfrac{\pi B}{2}-\dfrac{3\sqrt{3}A}{2}\right)\dfrac{\eta^{2}}{\xi^{4}}$	$+\dots$
$\Psi_4 =$						$\dfrac{2\pi\bar{D}\ln\xi}{3\xi^{3}}$	$+\dfrac{\pi D}{3\xi^{3}}$	$+\dfrac{3\bar{D}\eta(\ln\xi)^{2}}{\xi^{4}}$	$-\dfrac{(2\bar{D}-3D)\,\eta\ln\xi}{\xi^{4}}$	$-\dfrac{(D-3C)\,\eta}{\xi^{4}}$	$+\dots$
$\Psi_5 =$								$\dfrac{\text{constant}\times(\ln\xi)^{2}}{\xi^{4}}$	$+\dfrac{\text{constant}\times\ln\xi}{\xi^{4}}$	$+\dfrac{\text{constant}}{\xi^{4}}$	$+\dots$
Asymptotic form of	ψ_0	ψ_1	ψ_2	ψ_3^{*}	ψ_3	ψ_4^{*}	ψ_4	ψ_5^{**}	ψ_5^{*}	ψ_5	

TABLE 1. The matching scheme

9. Solution for f_2 and F_2

The third-order equation and boundary conditions for f_2 are

$$L_2(f_2) = S_2 = B_2 - 12f_0' - 2f_1'^2 + 4\eta f_0'' + 2\eta^2 f_0'^2 - \tfrac{32}{3}\eta\tau(3-\tau^2) + \tfrac{64}{3}\log\cosh\eta + \tfrac{16}{3}\tau^2 + \tfrac{104}{3}, \quad (9.1)$$

$$f_2(0) = f_2''(0) = 0. \quad (9.2)$$

The expansion (8.4) and the anticipated form of Ψ_2 show that to match $\xi^{-1}f_2$ we must have

$$f_2(\eta) \sim -\frac{3\sqrt{3}}{\pi} + O(1) = 1{\cdot}6540\eta + O(1) \quad \text{for} \quad \eta \gg 1. \quad (9.3)$$

The particular integral of (9.1) is odd, so that the appropriate fundamental solution of (9.1) is

$$u_{23} = u_2 - (2/\pi)\tan\tfrac{1}{2}\nu\pi\, u_3, \quad (9.4)$$

where $\nu = \frac{1}{2}(\sqrt{17}-1)$. Now (9.3) implies that $f_2'' \to 0$ as $\eta \to \infty$. Then (9.1) shows that $S_2 \to 0$ as $\eta \to \infty$ (the decay being exponential). Thus the matching condition (9.3) determines the constant B_2 which generates u_{24} ($= u_4$ with ν given the value just stated):

$$B_2 = -\tfrac{16}{3}(7 - 4\ln 2). \quad (9.5)$$

This guarantees that the correct multiple of u_{24} is being added to annul the coefficient of η^2 in the asymptotic form of the particular integral of the fourth-order equation for f_2. Numerical integration then shows that, for large η,

$$f_2 \sim -1{\cdot}6540\eta + 4{\cdot}1262 + O(e^{-2\eta}). \quad (9.6)$$

Thus

$$\psi_2 = \xi^{-1}f_2 \sim -1{\cdot}6540\xi^{-1}\eta + 4{\cdot}1262\xi^{-1} + O(\xi^{-1}e^{-2\eta}). \quad (9.7)$$

The potential-flow term $\xi^{-1}F_2$, which is zero on $\eta = \xi\sqrt{3}$ and matches the term $4{\cdot}1262\xi^{-1}$, is

$$\Psi_2 = \xi^{-1}F_2(\chi) = 4{\cdot}1262\,\frac{\xi - \eta/\sqrt{3}}{\xi^2 + \eta^2}. \quad (9.8)$$

10. Solution for f_3. The first inner eigenfunction

The procedure here is almost identical with that of the previous section. The third-order equation and boundary conditions for f_3 are

$$\begin{aligned} L_3(f_3) = S_3 = B_3 &- 20f_1' + f_1''f_2 - 5f_1'f_2' + 2\eta^2 f_1''' + 4\eta f_1'' + 2\eta^2 f_0 f_1'' \\ &+ 10\{\eta^2 f_0' f_1' - 2\eta[\tau - \tau^3 + 3\eta(\tau^2-1)^2]/\sqrt{3} \\ &+ 2[\eta(3\tau - \tau^3) - 2\ln\cosh\eta]/\sqrt{3}\}, \end{aligned} \quad (10.1)$$

$$f_3(0) = f_3''(0) = 0. \quad (10.2)$$

The Taylor expansion of (9.8) and (8.4) and the anticipated form of Ψ_3 show that, for matching $\xi^{-2}f_3$, we must have

$$f_3 \sim -4{\cdot}1262\eta/\sqrt{3} + O(1). \quad (10.3)$$

Because $\nu = 2$ when $n = 3$, we may find fundamental solutions of $M_3(f_3) = s_3$ with simple forms:

$$u_{31} = f_0', \tag{10.4}$$

$$u_{32} = f_0 - 2\eta f_0', \tag{10.5}$$

$$u_{33} = \eta^2 f_0' - \eta f_0 + 4, \tag{10.6}$$

$$u_{34} = 8\eta^3(1-\tau^2) - 12\eta^2\tau + 30\eta - \tfrac{117}{2}\tau. \tag{10.7}$$

The even solutions u_{31} and u_{33} are deleted as before. Although setting $B_3 = 30/\pi - 40(1+\ln 2)/\sqrt{3}$ ensures a linear asymptotic behaviour for f_3, the precise linear condition (10.3) cannot be imposed because u_{32} (whose coefficient a_{32} is still arbitrary) asymptotes to a constant rather than a linear form. This inadequacy is overcome by replacing $\psi_3 = \xi^{-2} f_3$ in (4.6) by

$$\psi_3^* + \psi_3 = \xi^{-2}(\ln \xi f_3^* + f_3).$$

Then
$$M_3(f_3^*) = 0, \quad f_3^*(0) = f_3^{*\prime\prime}(0) = 0. \tag{10.8}$$

A constant asymptotic form of f_3^* can be matched by introducing a term of $O(\xi^{-2}\ln\xi)$ in the outer expansion. The required function is

$$f_3^* = a_{32}^*(f_0 - 2\eta f_0') = a_{32}^* u_{32}. \tag{10.9}$$

An additional forcing term $-a_{32}^*(f_0 f_0'' + f_0'^2)$ now appears on the right side of (10.1), leading to a new solution f_3 with asymptotic behaviour

$$f_3 \sim (11{\cdot}530 + 0{\cdot}800 a_{32}^*)\,\eta + c^* + 2a_{32}. \tag{10.10}$$

The constant c^* depends on a_{32}^*, which for matching, by satisfying (10.3), must have the value $-17{\cdot}390$. One arbitrary constant a_{32} persists in the final form of the inner expansion as a coefficient of u_{32}, which we eventually identify as the first inner eigenfunction.

11. Extension of the results

The modified outer expansion is the imaginary part of

$$w = -\frac{4}{\sqrt{3}}\zeta_1 - \frac{3\sqrt{3}}{\pi}\ln\zeta_1 + \frac{8{\cdot}2524}{\sqrt{3}}\zeta_1^{-1} + \zeta_1^{-2}[B\ln\zeta_1 + A] + \ldots, \tag{11.1}$$

where $\zeta_1 = \zeta e^{\pm\frac{1}{3}i\pi}$ according as $-\sqrt{3} \leqslant \chi < 0$ or $0 < \chi \leqslant \sqrt{3}$. As defined, ψ vanishes on $\eta = \pm\xi\sqrt{3}$ and is anti-symmetric. The outer eigensolutions

$$r^{-k}\sin k(\pi - \theta), \quad k = 1, 2, \ldots,$$

represent mass multipoles in the z plane and lead to further logarithmic terms in the outer expansion. A non-periodic eigensolution $\pi - \theta$, allowing for production of mass at O, is ignored.

The method of Libby & Fox (1963) leads to the inner eigensolutions

$$\xi^{-(n-1)}E_n = \xi^{-(n-1)} f_0' \int_0^\eta \frac{P_\nu[\tau(\bar\eta)]\,d\bar\eta}{f_0'(\bar\eta)}, \quad \nu = 2j \ (j = 1, 2, \ldots). \tag{11.2}$$

Now $n = \frac{1}{2}\nu(\nu+1)$ with $j = 1$ implies $n = 3$, verifying that the first eigenfunction is indeed $E_3 = f_3$. This allows for the possibility of an origin shift h along Ox:

$$h\,\partial\psi_0/\partial x = \tfrac{1}{3}h\xi^{-2}(f_0 - 2\eta f_0') + o(\xi^{-2}), \tag{11.3}$$

which may be compared with $a_{32}u_{32}$. For each inner eigensolution an extra logarithmic term appears in the inner expansion.

12. Discussion

In viscous flow problems, it is usually necessary to divide the flow field into regions where various physical processes such as convection and diffusion play either an important or unimportant part. In this paper, two such regions for $r \gg 1$ have been discussed in great detail. For r sufficiently small, vorticity gradients are expected to be large, so that the diffusion process gives the model

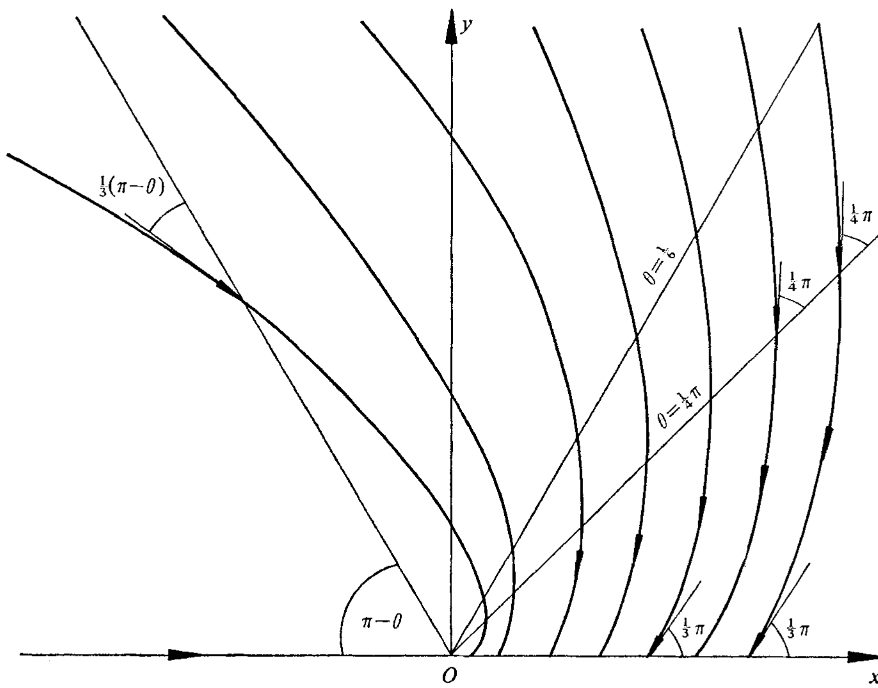

FIGURE 1. The potential-flow streamlines, Ψ_0 = constant, $r^{\frac{1}{3}}\sin\frac{1}{3}(\pi-\theta)$ = constant.

for this region. It seems unlikely that the description of the near field can be matched with that of the far field. One or more regions for intermediate values of r in which different combinations of terms come into balance may be required. In the transition region, the full Navier–Stokes equations may be needed to describe the flow. Further progress may be possible after making an Oseen-type linearization of the equations with respect to the composite stream function $\psi = 2\xi\tau - 2\eta/\sqrt{3}$, although in the outer region this is a forced rather than a forcing flow.

The flux integral condition (4.5) is obtained from the Navier–Stokes equations, which in turn are satisfied by the expansions. The asymptotic expansions in terms of large ξ should satisfy the flux condition to all orders, and this has been verified elsewhere up to terms of $O(\xi^{-1})$.

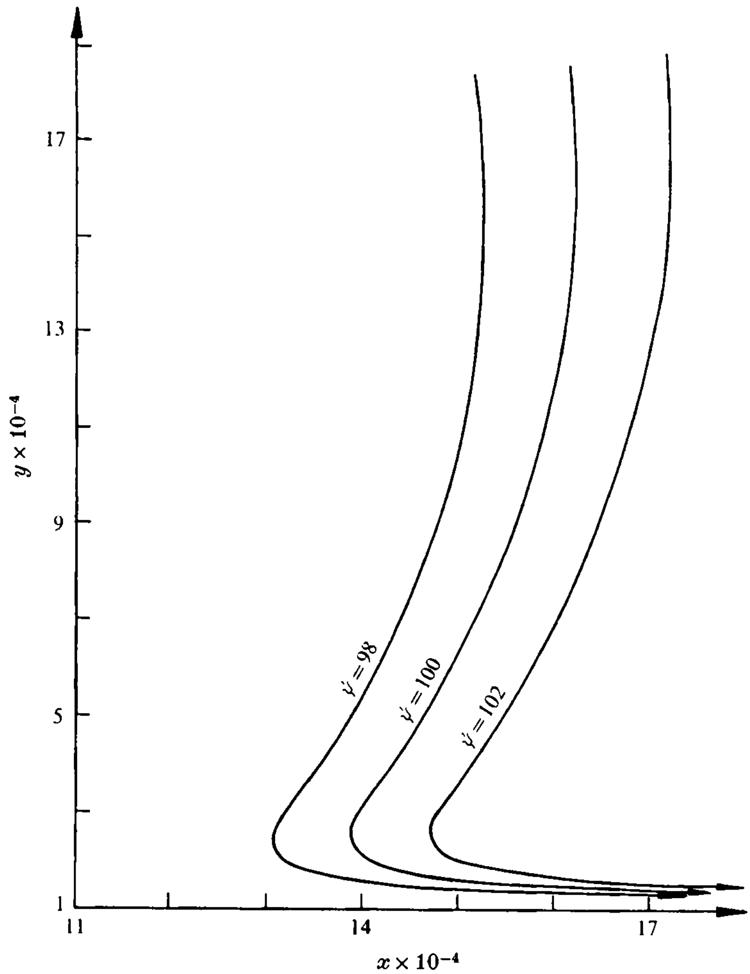

FIGURE 2. One-term composite-flow streamlines, $2\xi\tau - 2\eta/\sqrt{3}$ = constant, plotted in the z plane.

The leading terms of the radial velocity are of $O(r^{-\frac{1}{3}})$ in the jet and of $O(r^{-\frac{2}{3}})$ in the z plane. The lower order term overtakes the leading term when $\eta \sim \ln r$. Thus, in an intermediate region, the order decreases from $O(r^{-\frac{1}{3}})$ to that of the outer field velocity, namely $O(r^{-\frac{2}{3}})$. The dominant potential flow in the z plane for $r \gg 1$ is shown in figure 1. The streamlines cut the radial line θ = constant at the same angle $\frac{1}{3}(\pi - \theta)$; in particular, the angle of entry of all streamlines into the singular line $\theta = 0$ is $\frac{1}{3}\pi$. The direction of inflow seems to contradict the fact that the jet motion is in the positive-x direction. However, since this backflow is a second-order effect it is clearly permissible. The velocity field of the leading term $2\xi\tau - 2\eta/\sqrt{3}$ of a composite stream function expansion is shown in figure 2.

Since the solution depends crucially on the exponential decay of vorticity to all orders, it is worth noting that (3.5) may be obtained by replacing the condition $\Omega_0 \equiv 0$ by the less stringent one of constant pressure in the quiescent fluid. The vorticity ω_0 derived from (3.5) certainly approaches zero exponentially as $\eta \to \infty$. It then follows readily that Ψ_0 is indeed potential. By writing

$$\psi = \Psi_0 + \Psi = 2(\xi - \eta/\sqrt{3}) + \Psi,$$
$$\omega = \Omega_0 + \Omega = \Omega,$$

we obtain the linearized equation

$$\nabla^2\Omega + 2\Omega_\xi + (2/\sqrt{3})\,\Omega_\eta = 0. \tag{12.1}$$

Separable solutions which vanish on $\eta = \xi\sqrt{3}$ are found to have exponential decay as $\xi^2 + \eta^2 \to \infty$ with $0 < \eta/\xi < \sqrt{3}$. This supports the assumption of exponentially small vorticity at 'upstream infinity'.

As was anticipated in the introduction, the expansions are seen to contain eigenfunctions. The outer and the first inner eigensolutions have been interpreted physically. Higher order inner eigensolutions are more difficult to interpret but probably correspond to the possibility of having a more general distribution of circulation production in the finite part of the plane. The expansions satisfy the symmetry and mass requirements as well as the flux integral condition. Their structure has been determined beyond the terms discussed above (Capell 1971) and they appear to be correct.

The author thanks Professor A. F. Pillow for suggesting the problem and is grateful for both his and Professor K. Stewartson's advice during its solution.

Appendix

Provided that there are no mass sources, the momentum equation for incompressible flow may be written as

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \boldsymbol{\omega} + \nabla\left(\tfrac{1}{2}v^2 + \frac{p}{\rho}\right) - \nu\nabla^2\mathbf{v} = \mathbf{f}. \tag{A 1}$$

The force $\rho\mathbf{f}$ per unit volume, may be viewed as the momentum source density. When $\mathbf{v} = (v_1, v_2, 0)$ and $\mathbf{f} = f\mathbf{i}$, the x component of (A 1) may be written as the conservation result for x momentum:

$$\partial v_1/\partial t + \nabla . \mathbf{\Pi} = f. \tag{A 2}$$

A flux vector for x momentum is

$$\rho\mathbf{\Pi} = \left(\rho v_1^2 + p - 2\rho\nu\frac{\partial v_1}{\partial x}\right)\mathbf{i} + \left(\rho v_1 v_2 - \rho\nu\frac{\partial v_1}{\partial y} - \rho\nu\frac{\partial v_2}{\partial x}\right)\mathbf{j}. \tag{A 3}$$

For such two-dimensional flows, Helmholtz' vorticity equation, when written in the form

$$\frac{\partial \omega}{\partial t} + \nabla.(\omega\mathbf{v} - \nu\nabla\omega) = w = (\operatorname{curl}\mathbf{f})_3 = -\frac{\partial f}{\partial y}, \tag{A 4}$$

expresses conservation of circulation and displays the flux vector $\omega\mathbf{v} - \nu\nabla\omega$ for circulation as a sum of convective and diffusive components, w being the source density for circulation production. In the interior of a viscous fluid, under conservative body forces, w is of course, zero. Multiplication of (A 4) by y, and a little algebra, yields

$$\frac{\partial(\omega y)}{\partial t} + \nabla.\mathbf{V} = m = -y\frac{\partial f}{\partial y}, \tag{A 5}$$

$$\mathbf{V} = \omega y\mathbf{v} - \nu\nabla(\omega y) + \tfrac{1}{2}(v_1^2 - v_2^2)\,\mathbf{i} + v_1 v_2\mathbf{j}.$$

Then (A 5) is a conservation equation for the quantity $\iint_S \omega y\, dS$, which is the moment of circulation about Ox within a fixed arbitrary region S. The total rate of production of this quantity within S is

$$\begin{aligned}\iint_S m\, dS &= \iint_S -y\frac{\partial f}{\partial y}\, dS \\ &= \iint_S [-\nabla.(fy\mathbf{j}) + f]\, dS \\ &= \iint_S f\, dS,\end{aligned}$$

provided that $f = 0$ on the boundary ∂S of S. Under steady conditions, since both quantities are conserved and each is produced at a constant rate in S, the discharge through ∂S of moment of circulation is equal to the discharge of x momentum divided by density. It follows that, for steady flows, a source of x momentum of strength ρM at O (i.e. with $f = 0$ outside the origin) is equivalent to a circulation-producing dipole of strength M at O, the dipole axis being in the positive-y direction.

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